

F1 June 2018 (MA)

$$Q1) \sum_1^n r(r+3) = \sum_1^n r^2 + 3r = \sum_1^n r^2 + 3 \sum_1^n r$$

$$= \frac{n}{6}(n+1)(2n+1) + \frac{3n}{2}(n+1)$$

$$= \frac{n}{6}(n+1)[2n+1+9] = \frac{n}{6}(n+1)(2n+10)$$

$$= \frac{2n}{6}(n+1)(n+5) = \boxed{\frac{n}{3}(n+1)(n+5)}$$

$$Q2a) P = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

b) Enlargement by scale factor $k\sqrt{2}$ about $(0,0)$

$$c) PQ = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} k\sqrt{2} & 0 \\ 0 & k\sqrt{2} \end{pmatrix} = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}$$

$$\text{Area}_{\text{image}} = \text{Area}_{\text{object}} \times |\det(PQ)| \quad \Rightarrow$$

$$147 = 6 \times |\det(PQ)|$$

$$\therefore |\det(PQ)| = \frac{147}{6}$$

$$u^2 - (-u^2) = 2u^2 = (\det PQ) //$$

$$\therefore 2u^2 = \frac{147}{6}$$

$$u^2 = \frac{147}{12}$$

$$u = \pm \sqrt{\frac{147}{12}} = \pm \frac{7}{2}$$

$$u > 0 \quad \therefore \boxed{u = \frac{7}{2}}$$

Q3a)

$$y^2 = 6x$$

$$4a = 6$$

$$a = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \boxed{S\left(\frac{3}{2}, 0\right)}$$

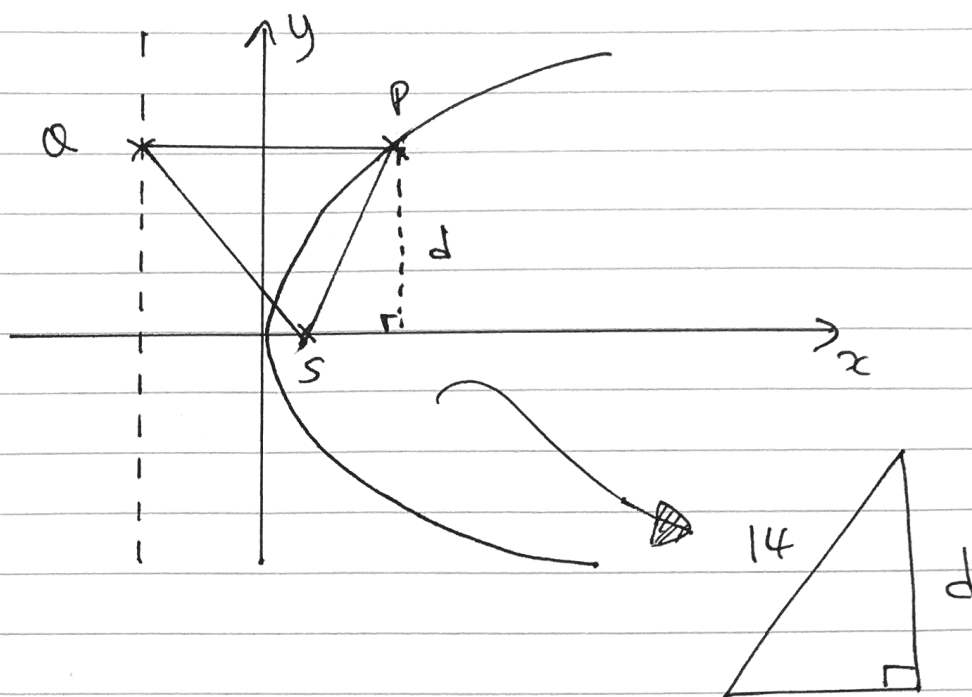
b) P will be equidistant from Q and S.

$$\therefore \boxed{PS = 14}$$

c) Directrix eqn: $x = -\frac{3}{2}$.

$$\therefore x \text{ coordinate of } P = -\frac{3}{2} + 14$$

$$= \frac{25}{2} //$$



$$\begin{aligned} \therefore d &= y\text{-coordinate of } P \\ &= \sqrt{14^2 - 11^2} = 5\sqrt{3} // \end{aligned}$$

$$\text{so } \boxed{P\left(\frac{25}{2}, 5\sqrt{3}\right)}$$

$$(Q4a) \quad A = \begin{pmatrix} 2p & 3q \\ 3p & 5q \end{pmatrix}$$

$$\begin{aligned} \det A &= 2p(5q) - 3p(3q) = 10pq - 9pq \\ &= pq // \end{aligned}$$

$$\therefore \boxed{A^{-1} = \frac{1}{pq} \begin{pmatrix} 5q & -3q \\ -3p & 2p \end{pmatrix}}$$

$$b) \quad XA = B$$

$$XAA^{-1} = BA^{-1}$$

$$X = BA^{-1}$$

$$\therefore X = \begin{pmatrix} p & q \\ 6p & 11q \\ 5p & 8q \end{pmatrix} \begin{pmatrix} 1 \\ pq \end{pmatrix} \begin{pmatrix} 5q & -3q \\ -3p & 2p \end{pmatrix}$$

$$X = \frac{1}{pq} \begin{pmatrix} 5pq - 3pq & -3pq + 2pq \\ 30pq - 33pq & -18pq + 22pq \\ 25pq - 24pq & -15pq + 16pq \end{pmatrix}$$

$$X = \frac{1}{pq} \begin{pmatrix} 2pq & -pq \\ -3pq & 4pq \\ pq & pq \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & -1 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}$$

● (Q5a) $(z^2 + 9)(z^2 + az + b) = z^4 + az^3 + bz^2 + 9z^2 + 9az + 9b$

compare coefficients with LHS of given eqn,

$$z^3: \boxed{a = -6} //$$

$$z^2: 9 + b = 34 \quad \therefore \boxed{b = 25} //$$

● b) $(z^2 + 9)(z^2 - 6z + 25) = 0$

$$z^2 + 9 = 0 \quad \therefore \quad z^2 = -9$$

$$z = \pm 3i$$

$$\therefore \boxed{\begin{matrix} z = 3i \\ z = -3i \end{matrix}}$$

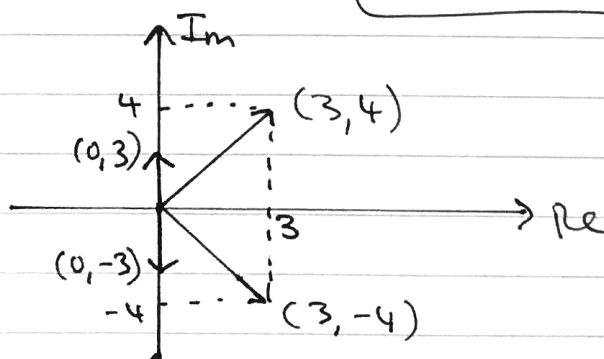
$z^2 - 6z + 25 = 0$: By Quadratic formula...

$$\left. \begin{matrix} a = 1 \\ b = -6 \\ c = 25 \end{matrix} \right\} z = \frac{+6 \pm \sqrt{36 - 4(1)(25)}}{2}$$

$$z = 3 \pm \frac{8i}{2} //$$

so

$$\boxed{\begin{matrix} z = 3 + 4i \\ z = 3 - 4i \end{matrix}}$$



$$\bullet \text{ (Q6a)} \quad f(x) = \frac{2x^3 + 6}{x^{\frac{1}{2}}} - 9$$

$$= 2x^{5/2} + 6x^{-1/2} - 9 //$$

$$\therefore f'(x) = 5x^{3/2} - 3x^{-3/2} //$$

$$f(0.45) = 0.21595\dots$$

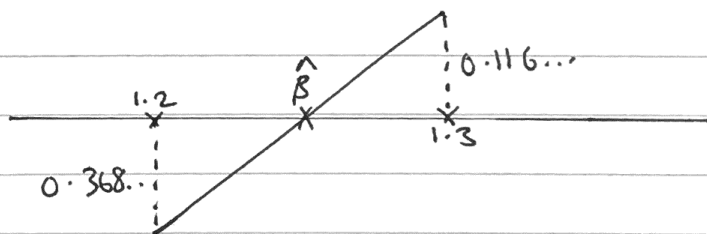
$$f'(0.45) = -8.4287\dots$$

$$\text{So } \hat{\alpha} = 0.45 - \frac{0.21595\dots}{-8.4287\dots}$$

$$= \boxed{0.476} \text{ to 3 dp}$$

$$\text{b) } f(1.2) = -0.36789\dots$$

$$f(1.3) = 0.116141\dots$$



$$\Rightarrow \frac{0.116\dots}{0.368\dots} = \frac{1.3 - \hat{\beta}}{\hat{\beta} - 1.2}$$

$$\text{Let } \frac{0.116\dots}{0.368\dots} = c,$$

$$\text{then } \frac{1.3 - \hat{\beta}}{\hat{\beta} - 1.2} = c$$

$$c\hat{\beta} - 1.2c = 1.3 - \hat{\beta}$$

$$\hat{\beta}(c + 1) = 1.3 + 1.2c$$

$$\therefore \hat{\beta} = \frac{1.3 + 1.2c}{c + 1} = \boxed{1.276}$$

to 3 d.p.

(7a)

$$5x^2 - 4x + 3 = 0$$

$$x^2 - \frac{4}{5}x + \frac{3}{5} = 0$$

$$\therefore \alpha + \beta = \frac{4}{5}$$

$$\alpha\beta = \frac{3}{5}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} \equiv \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} \equiv \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{\left(\frac{4}{5}\right)^2 - 2\left(\frac{3}{5}\right)}{\left(\frac{3}{5}\right)^2}$$

$$= \boxed{\frac{-14}{9}}$$

$$b) \left(x - \left(\frac{3}{\alpha^2}\right)\right) \left(x - \left(\frac{3}{\beta^2}\right)\right) = 0$$

expanding,

$$x^2 - x \left(\frac{3}{\beta^2} + \frac{3}{\alpha^2}\right) + \frac{9}{\alpha^2 \beta^2} = 0$$

$$x^2 - 3x \left(\frac{1}{\beta^2} + \frac{1}{\alpha^2}\right) + \frac{9}{\left(\frac{3}{5}\right)^2} = 0$$

$$x^2 - 3x \left(-\frac{14}{9}\right) + 25 = 0$$

$$x^2 + \frac{14}{3}x + 25 = 0$$

$$\underline{\times 3} : \boxed{3x^2 + 14x + 75 = 0}$$

$$(Q8) \quad \underline{n=1} : \begin{matrix} \text{LHS} \rightarrow \end{matrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^1 = \text{RHS} = \begin{pmatrix} a' & 0 \\ \frac{a-b}{a-b} & b \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

\therefore true for $n=1$.

now assume the given relationship is true for $n=k$,

$$\text{i.e. } \left[\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^k = \begin{pmatrix} a^k & 0 \\ \frac{a^k - b^k}{a-b} & b^k \end{pmatrix} \right]$$

consider $n=k+1$,

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^{k+1} = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^k \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^k & 0 \\ \frac{a^k - b^k}{a-b} & b^k \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^k(a) & 0 \\ \frac{a(a^k - b^k)}{a-b} + b^k & b^k(b) \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \frac{a^{k+1} - b^k a}{a-b} + \frac{b^k(a-b)}{a-b} & b^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ \frac{a^{n+1} - b^{n+1}}{a-b} & b^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ \frac{a^{n+1} - b^{n+1}}{a-b} & b^{n+1} \end{pmatrix}$$

\therefore true for $n = k+1$

So: true for $n=1$
true for $n=k+1$ when true for $n=k$.

\therefore By Mathematical Induction true for
all $n \in \mathbb{Z}^+$

Q9. a)

$$\frac{z - 4i}{z + 3i} = i$$

$$z + 3i^2 = z - 4i$$

$$z - 3 = z - 4i$$

$$z(i - 1) = 3 - 4i$$

$$z = \frac{3 - 4i}{-1 + i}$$

$$\therefore z = \frac{(3 - 4i)(-1 - i)}{(-1 + i)(-1 - i)} = \frac{-3 - 3i + 4i + 4i^2}{1 + 1}$$

$$z = \frac{-3 - 4 + (4 - 3)i}{2}$$

$$z = \frac{-3 - 4}{2} + \frac{(4 - 3)}{2}i$$

$$z = -\left(\frac{4 + 3}{2}\right) + \left(\frac{4 - 3}{2}\right)i$$

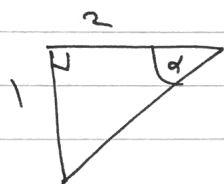
$$\bullet \text{ bi) } \underline{u=4} : z = -\frac{(4+3)}{2} + \frac{(4-3)}{2}i$$

$$z = -\frac{7}{2} + \frac{1}{2}i$$

$$\therefore |z| = \sqrt{\left(-\frac{7}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \boxed{\frac{5\sqrt{2}}{2}}$$

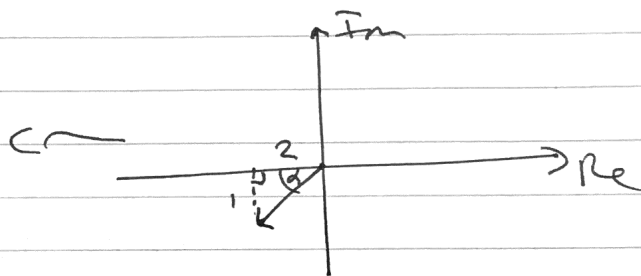
$$\bullet \text{ ii) } \underline{u=1} : z = -\frac{(1+3)}{2} + \frac{(1-3)}{2}i$$

$$z = -2 - i$$



$$\tan d = \frac{1}{2}$$

$$\therefore d = \arctan \frac{1}{2}$$



clockwise from real axis
is negative

$$\text{so } \arg z = -\left[\pi - \arctan \frac{1}{2}\right] = \boxed{-2.678}$$

Q10a)

$$xy = 144$$

$$P\left(12p, \frac{12}{p}\right)$$

$$y + x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

IMPLICIT
DIFFERENTIATION

$$\therefore \frac{dy}{dx} = -\frac{\frac{12}{p}}{12p} = -\frac{1}{p^2}$$

So at normal to P, $m = p^2$.

$$(p^2 \times \frac{-1}{p^2} = -1)$$

$$\Rightarrow y - \frac{12}{p} = p^2(x - 12p)$$

$$\Rightarrow y = p^2x - 12p^3 + \frac{12}{p}$$

b) $x=0$: $y = \frac{12}{p} - 12p^3$

$$\therefore R \left(0, \frac{12}{p} - 12p^3 \right)$$

$y=0$: $p^2x + \frac{12}{p} - 12p^3 = 0$

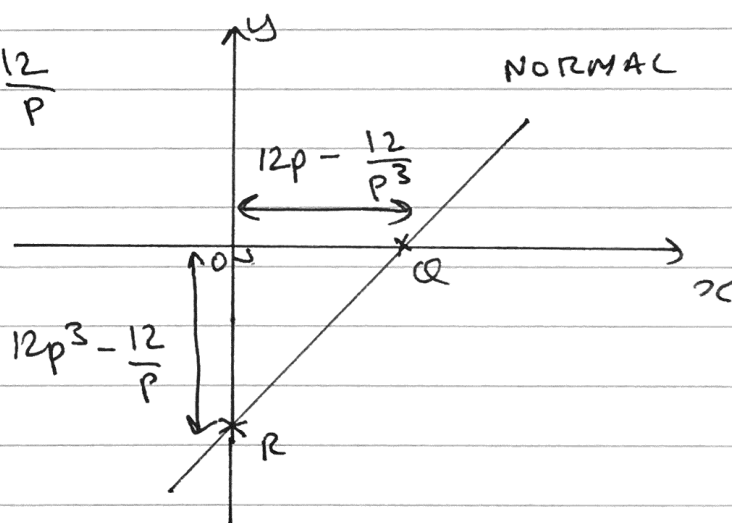
$$x = \frac{12p^3 - \frac{12}{p}}{p^2} = \frac{12p^3 - 12}{p^3}$$

$$= 12p - \frac{12}{p^3} //$$

so $Q \left(12p - \frac{12}{p^3}, 0 \right)$

c) length $OR = 12p^3 - \frac{12}{p}$
NOT $\frac{12}{p} - 12p^3$

as R is below the
 x -axis.



so Area $\Delta OQR = \frac{1}{2} \left(12p - \frac{12}{p^3} \right) \left(12p^3 - \frac{12}{p} \right)$
 $= \frac{12 \times 12}{2} \left(p - \frac{1}{p^3} \right) \left(p^3 - \frac{1}{p} \right) //$

$$\therefore 72 \left(p^4 - 1 - 1 + \frac{1}{p^4} \right) = 512$$

$$p^4 + \frac{1}{p^4} - 2 = \frac{512}{72} = \frac{64}{9}$$

$$\underline{\times 9p^4} : 9p^8 - 82p^4 + 9 = 0$$

$$(9p^4 - 1)(p^4 - 9) = 0$$

$$9p^4 - 1 = 0$$

$$p^4 = \frac{1}{9}$$

$$\therefore p = \pm \frac{1}{\sqrt{3}} //$$

$$p^4 - 9 = 0$$

$$p^4 = 9$$

$$\therefore p = \pm \sqrt{3} //$$

but note that we used $\left(12p^3 - \frac{12}{p} \right)$ as length OR as this was assumed to be positive since OR lies below the x -axis.

When using $p = \frac{1}{\sqrt{3}}$ or $p = -\sqrt{3}$ then

the expression $\left(12p^3 - \frac{12}{p} \right)$ evaluates to < 0 .

This would mean that R lies above the x -axis which is not true since we are told in the question that R is below the x -axis. So these values of p are invalid.

$$\therefore \boxed{p = \sqrt{3}} \quad , \quad \boxed{p = -\frac{1}{\sqrt{3}}}$$